Product-Mix Auctions: Examples, and Introduction to the PMA web app

PRELIMINARY DRAFT—please send comments and corrections to simon.finster@economics.ox.ac.uk or paul.klemperer@economics.ox.ac.uk.

Contents

0. Introduction

The following examples demonstrate the basic functions of the Product-Mix Auction (PMA) web app. In particular, they illustrate the benefits of combining auctions, as in the Bank of England's "ILTR" Auction or, e.g., the sale of related bonds. The examples are in individual *json*-format files here.
We encourage the reader to download them, and load them into the web app from local storage, and implement changes themselves (as described below); the companion slide set gives screenshots from these examples to assist the reader.

The web app is hosted at http://pma.nuff.ox.ac.uk.

Sections 1 and 2's examples illustrate how combining auctions for related goods
(i) allows the auctioneer to better express its preferences about how much of each good to sell as a function of the bids,
(ii) creates more competition between the bidders (hence better achieves the auctioneer's objectives, whether these be efficiency or profit maximisation or cost minimisation, etc.), and
(iii) optionally (see the last part of sec. 2) allows each bidder to express preferences about how the varieties of goods it will be allocated should depend upon the relative prices for the different varieties that the auction determines.

The software chooses prices and allocations based on predetermined rules. (It thus avoids any perception of favouritism or inappropriate use of discretion.) The software can also be used to test, and experiment with, different allocations.

Section 3 describes the Bank of England's auction. It also explains the distinction between the "horizontal" and "vertical" representations.

Section 4 gives more examples that might be particularly appropriate, for, e.g., the sale of related bonds.

Additional examples and options are illustrated in section 5, 6, 7, and an Appendix.

Notes
The bids for section 1,2,4, and the Appendix
are at prices for good 1: 17, 11, 9, 8, 7, 6
good 2: 17, 16, 15, 14, 13, 10
("Paired" bids are also added in some examples)
The bids for demoBoE, demoBoEstressed, and demoBoETQSS,
are based on the example figs1-2 in the paper "Product-Mix Auctions"
(on www.paulklemperer.org and at https://www.nuffield.ox.ac.uk/users/klemperer/productmix.pdf)

This is a preliminary draft—and some of the code is being developed further.
In particular,

- Some of the graphs may have slightly different labels on the screenshots than in the current version of the web app.
- Results are rounded to one decimal place in the web app, which can lead to apparently slightly inaccurate allocations (though this does not happen in any of the examples, below). Rounding precision can be adjusted in the full software.

Please send comments and corrections to simon.finster@economics.ox.ac.uk or paul.klemperer@economics.ox.ac.uk.

1. Independent Auctions

1.1 Auctioning a Single Variety

Load demoSingle
A single auction of 30 units; bids (17,11,9,8,7,6) on good 1 only
P=8; Q=30
Explain price and pictures

For most of the rest of the presentation we make all the bids the same size (11)—this makes checking results easier. (If you want, you can do this now by changing the numbers of units of bidder 1's bids from 4,8,5 to 11,11,11. Now P=9; Q=30; this is demoSingle2, and is one of the two separate auctions in the next subsection, 1.2.)

1.2 Two Separate Auctions

Load demoTwoSeparate
Two separate auctions of 30 units each
P=9,15; Q=30,30; *Bidder 1 now allocated some on both goods*

You may want to *explain the "Bids in price space" graph*
The black line segments meet at the pair of prices that the auction sets, and create three regions:
--any bid situated strictly within the bottom right region strictly prefers to be allocated good 1, and so will be allocated good 1
--any bid situated strictly within the top left region strictly prefers to be allocated good 2, and so will be allocated good 2
--any bid situated strictly within the bottom left (rectangular) region strictly prefers to be allocated nothing, and so will be allocated nothing
--any bid situated on a line segment is indifferent between the possibilities corresponding to the regions adjacent to the bid, and so will be allocated one of these (or a mixture of them)[1]
The current bids, each for a single good, are all on one or other axis.
We will later (end of Section 2, Section 5, and Appendix A.1) consider "Paired bids" that are offers to buy either good 1 or good 2, and specify a (maximum) price for each good—these bids are located away from the axes at the co-ordinates of the prices specified. We will also extend to more than two goods.

[1] [Note Q] The slope of the diagonal line segment assumes a bidder wishes to maximise the difference between her value and the price she pays. Alternative assumptions are possible—see section 7.4 for the case in which she wishes to maximise the ratio of value to price.

2. Combining the Auctions

We now combine the two auctions, as if the goods are identical from the auctioneer's point of view:

2.1 Introducing the Total Quantity Supply Schedule (TQSS)

Continue from section 1.2 (or Load demoTwoSeparate)
Change supply curves on both goods 1 and 2 to 60 units at price 0 (from 30 units at price 0)
Also enable Total Quantity Supply Schedule (TQSS) of 60 units at price 0 (demoCombine)
This TQSS constrains the total amount of all goods (good 1 plus good 2) allocated (here to 60). We will later use a different form of TQSS.[2]
P=13,13; Q=11,49

Explain why quantities more efficient [It is more efficient to accept bids of 14 and 13 on good 2 than 11 and 9 on good 1; total welfare has risen from 863 to 934[3]]

Explain prices [4] [We are finding the lowest prices that "makes everyone (bidders and auctioneer) happy to trade given their expressed preferences", that is, the lowest competitive-equilibrium prices. These are the lowest prices that the winners could have bid and still won--13 here on each good; if bidder 1 had bid less than 13 on good 1 she would have been displaced by bidder 4 on good 2.[5]]

2.2 Simple Examples

Continue from section 2.1 (or Load demoCombine)
P=13,13; Q=11,49

Auctioneer flexible about good 1, unwilling to sell more than 45 on good 2
(Auctioneer constrains quantity on good 2—*one inferior strategy*)
Change supply curve on good 2 to 45 units at price 0 (TQSS unchanged) (demoCombine2)
P=11,13; Q=15,45[6]

Auctioneer fully flexible, but costs her more (8 more) to sell good 2 than good 1
(Auctioneer constrains relative price on good 2—*another inferior strategy*)

[2] For this form of TQSS, we have a choice of "absolute" (which constrains the total amount allocated as some function that we can specify of the actual prices of the goods), or "normalised" (which constrains the total amount allocated as some function that we can specify of the "normalised" prices of the goods). Since we are just using a flat TQSS at zero until it jumps vertically to infinity, the choice is irrelevant here, since we will never allocate at prices below zero anyway.
This demonstration uses normalised (or no) TQSSs throughout, except in Section 3.2.
The "normalised price" is the price less the auctioneer's "marginal cost" (here 0), so is the auctioneer's "margin" at the auction prices. For an explanation of "normalised price" in a more general context, see note N (approx. 6).
[3] We measure social welfare conventionally, that is, as the sum of the auctioneer's profits and winning bidders' surpluses (the differences between their values and the prices they pay). Profits rose from 720 to 780; bidders' surpluses rose from 137 to 154.
[4] [Note P] When maximizing efficiency, the auction always finds the lowest possible competitive-equilibrium prices. That is, we find the lowest prices such that both the bidders and the auctioneer can be allocated precisely what they want at those prices, if their bids and supply curves reflect their actual preferences. So, for example, in the current case, charging a lower price for good 1 (11 rather than 13) would leave the bidders happy with the quantities they were allocated, but the auctioneer would then prefer to sell different quantities. Note that doing things this way (and not charging even lower prices where this would leave the bidders happy) also allows us to run auctions with multiple sellers, rather than with just a single auctioneer (though not in this app). (It is a property of the kind of bidding we permit, that if there are a large number of bids (strictly a continuum), and the "demand curves" they form are strictly downward sloping, there is a unique competitive equilibrium. Because bids are "lumpy", there may be multiple competitive equilibria, but there is always a unique lowest-price competitive equilibrium. That is, there cannot be two different equilibria, one of which has a strictly lower price on one good, and the other of which has a strictly lower price on a different good.)
[5] For a harder example, change supply curve on good 1 to 60 units at price 1 (i.e., raise the "reserve price" of good 1 to 1); results in P=14,13.
[6] [Note N] When we use a normalised TQSS, the program tells us the total quantity summed over goods (here 60), and the "normalised price" (here 11). **You do *not* need to understand the "normalised price" for our demonstration.** [It is "the value to the auctioneer of the ability to sell an additional unit", but respecting the constraints other than the TQSS (here, the supply curve constraints on the individual goods)". (Technically, it is the shadow price of the TQSS.) Here, the normalised price (11) is given by the auctioneer's "price-cost margin" (that is, the price she would receive minus her marginal cost of supply) on the next unit she would sell (namely good 1, since good 2 is constrained)]. See also note M (approx. 9).]

Change supply curve on good 2 to 60 units at price 8 (cf. marginal cost curves; so prices are 8 apart if other constraints don't bite--e.g., marginal cost becomes infinite at 60, but that constraint is unlikely to bite) (demoCombine3)
P=8,16; Q=38,22[7]

Load demoUpwardSloping
Auctioneer uses an upward sloping cost curve for good 2
rather than either fixed quantity or fixed relative price—*a better strategy!*
P=9,14; Q=24,36

Reasons why this is better:
--can better represent auctioneer's preferences/costs, so is more efficient;
--protects the auctioneer against calibration error (e.g., if it constrains relative prices, but misjudges the relative values of the goods it is selling, it can lose a lot of money if the bidders are better informed);
--is a bit like a trader in the market who responds to getting more orders on a stock by raising the price on the stock—the auction is just doing this automatically;
--makes collusion, and exercising market power, harder[8]

[Welfare (as conventionally measured) is now 806, compared with 761 at the beginning of the section, with 781 from the first "inferior strategy" (constraining quantity) and with 780 from the second "inferior strategy" (constraining relative prices), assuming that the current supply curves reflect the auctioneer's actual preferences.]

You may want to look at some of the graphs:
"Supply curve and unadjusted demand for good 1" and "Supply curve and unadjusted demand for good 2" show the "demand" (i.e., bids in order of price—the length of the steps are the sizes) and the supply curves.
At the chosen allocation, the auctioneer's margin (that is, the price she receives on the good minus her marginal cost of supply) is 9 for each good.[9]
The "TQSS and (normalised) total demand" compares the margin that is achievable for the best possible allocation with the TQSS constraint (i.e., we could draw a margin *versus* allocation graph for each good, and summing these graphs horizontally would create the red "(normalised) total demand" line.)[10]

Example of a paired bid
Add an additional bid to bidder 1 for 2 units, at prices (13, 17) (demoPaired)
P=9,14; Q=24,36 (as before)
Observe that the paired bid (which we made for just 2 units) is allocated good 1, even though the bidder bid a *lower* price for that good.
The reason is that this good is "better value" for it, assuming the prices bid reflect the values of the goods to the bidder. That is, bid price *minus* auction price is higher for good 1 (13-9=4) than for good 2 (17-14=3).[11]

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[7] See note P (approx. 4) – in this case, charging a lower price for good 2 (15 rather than 16) would leave the bidders happy with the quantities they were allocated, but the auctioneer would then prefer to sell different quantities.
[8] Of course, even linking the auctions without having upward sloping curves (as in demoCombine) reduces market power by increasing the numbers of bidders competing with each other, and so also increases the complexity of the (tacit or explicit) agreements bidders would need to make among themselves in order to collude. However, upward sloping curves greatly increase the complexity of the necessary agreements, and of exercising market power, especially if (as, for example, in the Bank of England's case) these curves are not revealed to the bidders.
[9] [Note M] The auctioneer's margins would always be the same on every good (and also equal to the "normalised price"—see note N (approx. 6)) if there were no "lumpiness" in the bids, so that the margins were continuous functions of quantities.
[10] Notes: 1.The graphs are a bit different if the TQSS is absolute rather than normalised.
2. Constructing the graphs is less straightforward when paired bidding is included, since the allocations on the two goods interact with each other.
[11] For example, the two goods might be a bottle of red wine and a bottle of white wine; see note Q (approx. 1).

3. The "Vertical" Representation, and the Bank of England's Auction

You can skip this section if you are only interested in "horizontal" problems

3.1 Vertical Representation

Up to now we have treated the two goods in the same way, i.e., "horizontally" (so we have input information about the two goods in the same way). However, if (as here) selling good 2 is more expensive than selling good 1 for the auctioneer, and good 2 is also worth more than good 1 to the bidders, it is natural to think of these goods as being ordered "vertically" -- this is the Bank of England's representation of their original problem in which the goods 1 and 2 corresponded to loans against strong and weak collaterals, respectively.[12]

Optionally, to see how the Bank of England implements this:
Load demoPaired, or continue from before
Alternative "vertical" representation of the final example of section 2
Change supply ordering to vertical
Also disable TQSS (since supply curve on good 1 does that job in the basic vertical representation) (demoVertical)
NO CHANGE (except to graphs): P=9,14; Q=24,36
(The interpretations of the graphs are discussed in Section 3.2 and Appendix A.2 and may be better skipped here.)

3.2 The Bank of England's Indexed Long-Term Repo Auction

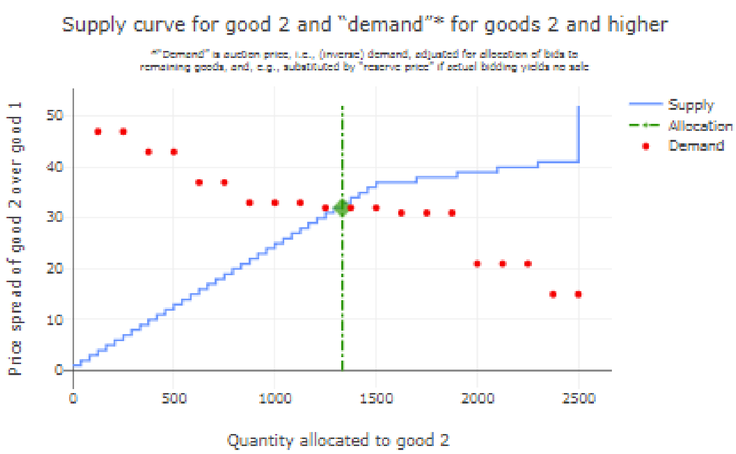
Load demoBoE
This is the example in figure 2 of the paper "Product-Mix Auctions" (at www.paulklemperer.org and at https://www.nuffield.ox.ac.uk/users/klemperer/productmix.pdf)
--it corresponds closely to the Bank of England's initial implementation; the data is test data that was provided by the Bank of England.
P=565,592; Q=1375,1125
The graph "Supply curve, and demand, for good 2" is similar to the kind of graph that appeared on the Bank of England's screens in their initial implementation. It illustrates clearly how the auction is finding competitive equilibrium. (Visualisations are less simple now that the Bank auctions more than 2 varieties at once.)

[12] [note V]. That is, the Bank of England's original implementation was concerned with the total amount, Q1+Q2, it allocates in loans, and the amount, Q2, it allocates against "weak collateral". So it was natural to write its total cost of supply as F1(Q1+Q2) + F2(Q2), for some functions F1, F2. By contrast, total cost in a "horizontal" auction of, e.g., two substitute bonds would be G1(Q1) + G2(Q2), for some functions G1, G2.
In simple cases (such as this example), vertical and horizontal representations can be mapped into each other (here, by writing F1 $\equiv$ G1 $\equiv$ 0, and F2 $\equiv$ G2). The Bank of England now allocates loans against more alternative qualities of collateral. See Appendix A.2 for discussion of the case of more than two goods.

(We can zoom in to the part of the picture we want to examine (*by creating box; double click to de-zoom*)
The graph corresponds to figure 2 in the paper. [The vertical axis shows price difference in basis points (fig 2 in the paper shows price difference in percentage); The horizontal axis shows millions of pounds of loans allocated to good 2 (fig 2 shows percentage of the 2500 available allocated to good 2).]
We can always generate a graph of this kind by choosing a flat supply curve for good 1 (2500 units at 0), and an appropriate supply curve for good 2 – we approximated the piecewise linear supply curve from fig 2 in the paper by 41 steps. (Using more steps would approximate it more closely.)
The "demand" curve tells us what the price spread (as a function of Q) would have to be if we were to allocate exactly Q units to good 2.
(So unless demand is very light, (2500-Q) units will be allocated to good 1, since the first supply curve permits a total of 2500 units at zero price.)[13]

Load demoBoEstressed. *Comparing this scenario (in particular the graph below) with the previous one (the graph above), illustrates the appeal of the auction—the comparison shows clearly how both relative price and quantities respond in the competitive equilibrium (hence in the auction outcome) to more aggressive bidding*.

This is the example in the paper, but increasing the prices on the bids for good 2:
P=565,597; Q=1167,1333
This corresponds to markets being more stressed: bidders are willing to pay higher interest rates to borrow against weak collateral (good 2), so the "relative demand curve" for good 2 (the red dots) is pushed out. (Note that the vertical axis has been rescaled slightly.)
With an upward-sloping supply curve, this results in allocating less of good 1 and more of good 2. The difference in prices between the goods has risen from 27 to 32 basis points.



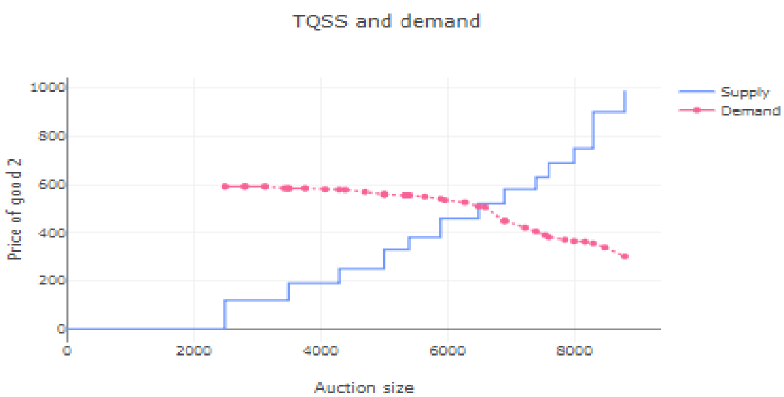
[13] The graph "supply curve for good 1 and demand for goods 1 and higher" has a similar interpretation, but is less useful here.

Load demoBoETQSS

The current auction (which we implemented in 2014, with the help of Elizabeth Baldwin) expands the auction size when the bidding is more aggressive.
In demoBoETQSS we have started with demoBoE, but enabled the TQSS so additional units (beyond the initial 2500) can be allocated as the price of one or more goods rises (in this example the price of good 2). (We have also added some additional bids)
The TQSS we use for “vertical” goods specifies increments in the total supply above the original “auction size” of 2500 (the length of the lowest vertical supply curve), and then scales all the supply curves in proportion to the auction size. (We typically use a different form of TQSS for “horizontal” goods—see Sections 2, and 4)
Note the new supply curve for good 2--the blue line in the higher graph below-- looks like the previous one except that the horizontal scale runs to 6500 rather than 2500.
The blue line on the “TQSS and demand” graph is the TQSS itself, showing the price of (in this example) good 2 as a function of the total auction size.

Supply curve for good 2 and “demand”* for goods 2 and higher

TQSS and demand



4. More Examples of "Horizontal" Applications, e.g., Related Bonds

The examples in section 3 were natural for "vertically-differentiated" goods (such as vintage- and non-vintage champagne). So we now give some more natural examples for "horizontally-differentiated" goods (such as red and white wine).

Load demoCombine
Auctioneer has some flexibility, but not complete flexibility.
e.g., she will sell 25 of good 1, 25 of good 2, and 10 of whichever of good 1 and good 2 gets best value
Change supply curves on both goods 1 and 2 to 35 units at price 0 (from 60 units at price 0; TQSS unchanged)
(demoBonds)
P=9,14; Q=25,35.
[Social welfare (as conventionally measured) is now 891—intermediate between the cases of two separate auctions (863), and the "fully-flexible" demoCombine from Section 2 (934). (Profits now =715; bidders' surpluses =176.)]

A more sophisticated auctioneer might only be willing to sell the extra 10 if she is getting especially good value
Change TQSS to 50 units at price 0 (from 60 units at price 0)
Also add two additional steps on the TQSS, the first of 5 units at 5, and the second of 5 units at 10
(demoBonds2)
P=10,14;[14] Q=22,35

Auctioneer might prefer to sell good 1 than to sell good 2—e.g., she requires a premium of 5 per unit to sell good 2.
Auctioneer might also prefer to *not* sell more than 20 of either good—e.g., she requires a premium of 5 per unit to sell more than 20 of good 1, and a premium of 3 per unit to sell more than 20 of good 2.
Change good-1 supply curve to 20 units at price 0 (from 35 units at price 0)
Also add an additional step on good-1 supply curve of 15 units at 5 (the "more-than-20" premium)
Change good-2 supply curve to 20 units at price 5 (from 35 units at price 0) (the "good-2" premium)
Also add an additional step to good-2 supply curve of 15 units at 8 (the "good-2" + "more-than-20" premia)
(demoBonds3)
P=11,14; Q=20,35

For the merits of having upward-sloping supply curves and a TQSS, and for discussion of the graphs,see Section 2.2 (after "load demoUpwardSloping") [15]

[14] See Note P (approx. 4) – in this case, charging a lower price for good 1 (9 rather than 10) would leave the bidders happy with the quantities they were allocated, but the auctioneer would then prefer to sell different quantities (Q=20,35)
[15] In this case, at the chosen allocation, the auctioneer's margin is 6 for each good.

5. More Examples of Paired Bids

(We continue to illustrate using the "horizontal" case—see the Appendix for the "vertical" case)

load demoPairedHoriz
OR continue from section 4 (demoBonds3) and add an additional bid to bidder 1 for 2 units, at prices (15, 17)
P=11,14; Q=20,35 (as in demoBonds3 in Section 4)
Observe that the paired bid (which we made for just 2 units) is allocated good 1, even though the bidder bid a *lower* price for that good.
The reason is that this good is "better value" for it, assuming the prices bid reflect the values of the goods to the bidder. That is, bid price *minus* auction price is higher for good 1 (15-11=4) than for good 2 (17-14=3).[16]

Change the paired bid to prices 15, 18 (still 2 units) (demoPairedHoriz2)
P=11,14; Q=20,35 (as before)
Observe that the bid is now indifferent between the goods at the auction prices—it sits exactly on the diagonal line in the "bids in price space" graph. The bid can therefore be allocated as suits the auction, which turns out to be good 2, as can be seen from the "bid allocations" table, and also in the "allocation of bids to goods" graph.

Change the paired bid to 5 units (still same prices 15, 18) (demoPairedHoriz3)
P=11,14; Q=20,35 (as before)
The bid is still indifferent between the goods at the auction prices (and still) sits exactly on the diagonal line in the "bids in price space" graph.
But the auction has now rationed it between good 1 (3 units) and good 2 (2 units), as can be seen from the "bid allocations" table.
In the "allocation of bids to goods" graph, it is shown in both columns as a white circle, indicating that it has been rationed.[17]

Change the paired bid to 50 units at prices 12, 15 (demoPairedHoriz4)
P=12,15; Q=20,35
The bid is now indifferent between three options: both goods and also no sale. It suits the auction to ration it between all three of these options.[18]

[16] See note Q (approx. 1).
[17] [Note G] In the graph "Supply curve and unadjusted demand for good 2" the allocation is at a price below the (unadjusted) demand curve – this is because the paired bid is included in both the graphs "Supply curve and unadjusted demand for good 1" and "Supply curve and unadjusted demand for good 2". Including paired bids will often create prices below the unadjusted demand for this reason (though the two previous paired bid examples were constructed to avoid this issue).
[18] [Note R] When the auction is indifferent about which bid to allocate goods to, the program rations to be "fair" between bids, to the extent possible. [You might ask what "fair" means if some of the "tied" bids are "paired" (e.g., one of two bids "tied" on one good is a "paired" good, already partially filled on another good). Complications like these don't arise in our examples. The basic version in the web app rations to favor paired bids, but we offer several options in the "full-featured" and the offline versions. (Note that a bidder who bids her actual values should be indifferent among all the options, since her bids can only be rationed if she is indifferent about whether they are accepted at the prices the auction sets.)]

6. More Than Two Varieties

(We continue to illustrate using the "horizontal" case—see the Appendix for the "vertical" case)

Load demoMoreHorizGoods
Now selling 4 goods; bidder 8 has complex demand for all 4 goods
P=15,15,16,17; Q =11,28,10,11[19]

With 3 goods we have a 3D-graph of "Bids in price space"

Load demo3GoodsHoriz--This takes a little longer to run (because of the graph)
P=12,8,7; Q=10,10,10
You can rotate this graph to view it from any angle; it is a useful tool for understanding the auction
Looking from the origin, bids to the right of the pink, to the right of the violet, and below the red surface (i.e, at relatively high prices on the X-axis) are allocated good 1; bids to the left of the blue, to the left of the violet, and below the green surface (i.e., bids at relatively high prices on the Y-axis) are allocated good 2; bids above the yellow, green, and red surface (i.e, at relatively high prices on the Z-axis) are allocated good 3; bids to the left of the pink, to the right of the blue, and below the yellow surface are allocated nothing.
Note that bidder 6's bid of size 5 (but rationed to 2 units and so labelled 2 in the graph) is indifferent between good 2 *and* good 3 *and* no purchase at the prices of goods 2 and 3 (8 and 7, equalling her bid prices); bidder 4's bid (of size 9) is indifferent between goods 2 and 3 at these prices given her bid prices, 12 and 11 (because 12 – 8 = 11 – 7).

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[19] Prices for good 1 and 2 are now both set by the marginal bid on good 2 (bid 2 of bidder 1 for 15).
The price for good 3 is set by bidder 8's marginal bids (bid 6 and 7 for 16).
The price for good 4 is set by the supply curve of good 4 in conjunction with the "normalised price" (price *minus* supply curve) of 15 across all goods. The price of good 4 is also compatible with the paired bids, e.g., bid 8 of bidder 8 is allocated good 4 so has to be weakly better off receiving good 4 instead of 3. (Note G (approx. 17) explains why the allocation is at a price below the (unadjusted) demand curve in the graph "Supply curve and unadjusted demand for good 4".)

7. Some of the Further Options

7.1 Pay-As-Bid Pricing

We can do pay-your-bid pricing

This demo was based on *uniform pricing* (on any given good, each winning bid is filled at the same price).
We can also easily do *pay-your-bid pricing* (i.e., "discriminatory" or "multi-price" pricing—every winning bid pays the price actually bid. (This is not in the web app yet.)
We prefer uniform pricing, especially with paired bids, since honest bidding is usually best for bidders in uniform.[20]
So the bidding is easy in a uniform-price auction; so entry by inexperienced bidders is easier than in a pay-your-bid auction, and the outcome is generally more efficient than in a pay-your-bid auction, and the pattern of bids gives the auctioneer better information than in a pay-your-bid auction.

7.2 General Trade-offs between Goods, and Additional Constraints

Thee "Full-Featured" version of the web app (drop-down menu in the top left corner) permits more general trade-offs between goods, and additional constraints to individual bids

load demoGenTradoffs
Bidder 1's bid is for an overall quantity of 10 [i.e., for up to a total of 10 units].
It specifies a trade-off 4:5, and specifies a maximum quantity of 8 for good 1 (up to price 5) and a maximum quantity of 6 for good 2 (up to price 8).
That is: bidder 1 bids for "*either* {up to 8/4 = 2 units of good 1 at up to price 5} *or* {up to 6/5 = 1.2 units of good 2 at up to price 8} *or* {nothing if prices are too high}",
subject to the constraint $4\times(\text{Quantity of good 1}) + 5\times(\text{Quantity of good 2}) \le 10$.[21]

Bidder 2's bid is for an overall quantity of 10.
It specifies a trade-off 2:1, and specifies maximum quantities of 10 for both good 1 and good 2.
That is, bidder 2 bids for "*either* {up to *10/2* = 5 units of good 1 at up to price 7} *or* {up to 10/1 = 10 units of good 2 at up to price 5} *or* {nothing if prices are too high}",
subject to the constraint $2\times(\text{Quantity of good 1}) + 1\times(\text{Quantity of good 2}) \le 10$.
[Note that since the maximum quantities specified for good 1 and good 2 (both 10) equal the overall quantity specified (also 10) they impose no constraints on the individual quantities of good 1 or good 2 beyond those imposed by the overall constraint ($2\times(\text{Quantity of good 1}) + 1\times(\text{Quantity of good 2}) \le 10$) .]

Both bidder 3's bids are for an overall quantity of 30 units, specifying trade-off 1, and a maximum quantity of 30.
So both these bids correspond to the simple standard bids used in the standard version of the App (see previous secs).

P = 4,4; Q = 30,30[22]

We see bidder 1 receives 1 unit of good 1, and 1.2 units of good 2 (he prefers good 2, but her quantity constraint on good 2 is binding), and her overall quantity constraint is binding, i.e. her bid is fully allocated ($4\times1 + 5\times1.2 = 10$).

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[20] e.g., in demoSingle, only bidder 3 (who bid 8) can potentially gain by distorting (she would like to bid just above 7; if she's forced to bid an integer, she can bid 7, which lowers the price she pays to 7, but she is then rationed to 4.8).

[21] In general, with M:N trade-offs, an overall bid quantity of Q, and maximum constraints of Q1 and Q2 on the quantities of good 1 and good 2, respectively, a bid is for {up to Q1/M units of good 1 at up to P1} *or* {up to Q2/N units of good 2 at up to P2} *or* {nothing if prices are too high}, subject to receiving a maximum of Q units in total, *relative to the specified trade-off*, i.e., respecting the constraint $M\times(\text{Quantity of good 1}) + N\times(\text{Quantity of good 2}) \le Q$.

[22] Using M:N trade-offs instead of 1:1 can amplify the effects of rounding (we are currently rounding all output to one decimal place in the web app). For example, changing bidder 1's constraint on good 2 from 6 to 7 results in an allocation to her of 0.75 of good 1 which, when rounded to 0.8, seems to violate her overall quantity constraint ($4\times0.8+5\times1.4 = 10.2 > 10$). Additionally changing her tradeoff from 4:5 to 40:5 amplifies the apparent violation. (She is now allocated 0.075 (rounded to 0.1) of good 1, and $40\times0.1+5\times1.4 = 11 > 10$.)

Bidder 2 is allocated 5 units of good 1, and nothing of good 2, and his bid is also fully allocated (2×5 = 10).

We can constrain the auction in other ways, e.g., we can add constraints across sets of bids (this is available in the full software, but not in the web app).

7.3 Profit Maximisation

We can do profit maximisation rather than efficiency[23]

Reload demoSingle
Change maximisation strategy to maximise profitability (demoProfitmax)
P=8; Q=30 –no change, but note that auctioneers profit =240 (=30×8)
Now change supply curve price to 5 (demoProfitmax2)
P=9; Q=23 –auctioneers profit = 92 [=23×(9-5);better than 30×(8-5)=90]
Now change supply curve price to 7 (demoProfitmax3)
P=11; Q=12 -- auctioneers profit = 48 [=12×(11-7); better than 23×(9-7)=46]
[Observe that in the latter two cases, maximising efficiency would yield P=8, Q=30 , as before.]

We can do much more with profit maximisation (not all in the web app yet).

7.4 Budget-Constrained Version: Iceland's Auction

We can do a budget-constrained version, as originally announced, and planned for, by the government of Iceland (but was later abandoned in their political crisis)

Load demoIceland
Auction chooses prices to maximise auctioneer's profits, subject to allocating each bid the good that is best for it at the auction prices (and for each bid, the sum across goods of (auction price x allocated quantity) does not exceed budget submitted in the bid).
P=1,5; Q=0+6+3+1=10,1.4+0+0+0.4=1.8

Note that the budget constraint means that the good that is "best value" for a bid is the good that has the highest *ratio* of bid price to auction price.[24] For example, bid 2 (at prices 2,7) strictly prefers (and so is only allocated) good 1 at the auction prices (1,5), because 2/1 > 7/5. (The bid's surplus from spending its budget, b, on buying good 1 is (b/1).(2-1) = b.(2/1) - b; its surplus from spending b on good 2 is (b/5).(7-5) = b.(7/5) – b.)
(As usual, we assume any bid accurately reflects a bidder's true preferences.)

7.5 "Positive and Negative Dot-Bids" Version

This extends the standard version to permit all participants (buyers and seller) to express *any* strong substitute preferences for indivisible goods.

There are other options in the web app.

Our offline software offers yet more options, and flexibility.

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[23] More efficient allocations are often more profitable, but they needn't be. For example, in demoTwoSeparate, the two completely separate auctions of 30 units yielded P=9,15; Q=30,30; so revenue = 720, but changing to a more–efficient completely flexible combined auction of 60 units yielded P=13,13; Q=11,49; so revenue = 750. However, in the case we considered next, in which the auctioneer had some, but not complete, flexibility, so had to sell at least 25 of each good, and had flexibility only on the final 10, we obtained P=9,14; Q=25,35; so efficiency was higher than with two completely separate auctions of 30 units each, but revenue was lower (715).

[24] Recall that, by contrast, without budget constraints "best value" for a "paired bid" is the good that has the highest bid price *minus* auction price (see section 2).

Appendix—Additional Examples for the Vertical Case

A1. More Examples of Paired Bids

Load demoPairedVert
P=9,14; Q=24,36 (as before)
Change the paired bid to prices 13, 18 (still 2 units)
P=9,14; Q=24,36 (as before)
Observe that the bid is now indifferent between the goods at the auction prices—it sits exactly on the diagonal line in the "bids in price space" graph. The bid can therefore be allocated as suits the auction, which turns out to be good 2, as can be seen from the "bid allocations" table, and also in the "allocation of bids to goods" graph.
Change the paired bid to 5 units (still same prices 13, 18)
P=9,14; Q=24,36 (as before)
The bid is still indifferent between the goods at the auction prices (and still) sits exactly on the diagonal line in the "bids in price space" graph.
But the auction has now rationed it between good 1 (2 units) and good 2 (3 units), as can be seen from the "bid allocations" table. In the "allocation of bids to goods" graph, it is shown in both columns as a white circle, indicating that it has been rationed.
Change the paired bid to 50 units (still same prices 13, 18)
P=13,18; Q=24,36 (as before)
The bid is now indifferent between three options: both goods and also no sale. It suits the auction to ration it between all three of these options (see note R (approx. 18)).

A2. More Than Two Varieties

Load demoMoreVertGoods
We are now selling 4 vertical goods: P=15,17,20,23; Q=12,10,9,11
In the "vertical" representation, the *n*th supply curve represents the incremental cost of selling good *n* and higher rather than good *n*-1.[25] (So, for example, the supply curve for good 1affects the auctioneer's "marginal cost" on all the goods.)
The auction outcome is illustrated in the 4 graphs of "Supply curve and 'demand' for goods *n* and higher", for *n*=1,...,4.
Observe 42 units of good 1 & higher are sold, so the "marginal cost" of good 1 is 15, which sets the price of good 1.
For good 2, 30 units of good 2 & higher are sold, so the incremental marginal cost of selling good 2 over the marginal cost of selling good 1 is 2, so the total marginal cost of good 2 is 2+15=17, which sets the price of good 2.
For good 3, 20 units of good 3 & higher are sold, so incremental marginal cost of good 3 is 2, and its total marginal cost is 2+17=19. However, the highest unfilled bid on good 3 has a price of 20, so price of good 3 is 20.
Since 10 units of good 4 are sold, its incremental marginal cost is 3 (and its total marginal cost is therefore 19+3=22), so the auction price is 3+20 (3+the price of good 3) = 23.
Notice that the auctioneer need not, of course, (and does not here) sell all the 60 units that the supply curve permits.

With 3 goods we have a 3D-graph of "Bids in price space"

Load demo3GoodsVert--This takes a little longer to run (because of the graph)
P=6,8,12; Q=10,11,9
You can rotate this graph to view it from any angle; it is a useful tool for understanding the auction
Looking from the origin, bids to the right of the pink, to the right of the violet, and below the red surface (i.e, at relatively high prices on the X-axis) are allocated good 1; bids to the left of the blue, to the left of the violet, and below the green surface (i.e., bids at relatively high prices on the Y-axis) are allocated good 2; bids above the yellow, green, and red surface (i.e, at relatively high prices on the Z-axis) are allocated good 3; bids to the left of the pink, to the right of the blue, and below the yellow surface are allocated nothing.
Note that bidder 5's bid of size 4 (but rationed to 2 units and so labelled 2 in the graph) is indifferent between good 1 *and* good 2 *and* no purchase at the prices of goods 1 and 2 (6 and 8, equalling her bid prices); bidder 4's bid (of size 9) is indifferent between goods 2 and 3 at the auction prices,6 and 8, given her bid prices, 10 and 12 (because 10– 6 = 12– 8).

[25] So with 4 goods, the auctioneer's total cost of supply in a "vertical" auction in which she sells Qi of good i is F1(Q1+Q2+Q3+Q4) + F2(Q2+Q3+Q4) + F3(Q3+Q4) + F4(Q4), for some functions Fi, i=1,…,4. Her total cost in a "horizontal" auction in which she sells Qi of good i is G1(Q1) + G2(Q2) + G3(Q3) + G4(Q4), for some functions Fi, , i=1,…,4. In simple cases (such as the examples of section 2), vertical and horizontal representations can be mapped into each other. (See note V (approx. 12).)